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TOPOLOGICAL FIELD THEORY AND NONLINEAR σ-MODELS ON SYMMETRIC SPACES

L. Martina [1] , *O. K. Pashaev*[†∗ 2] *and G. Soliani* [3]

Dipartimento di Fisica dell'Università and INFN Sezione di Lecce
73100 Lecce, Italy
†) *Joint Institute for Nuclear Research, 141980 Dubna, Russia* [4]
∗) *Department of Mathematics, TUBITAK-Marmara Research Center*
41470 Gebze-Kocaeli, Turkey

Abstract

We show that the classical non-abelian pure Chern-Simons action is related to nonrelativistic models in (2+1)-dimensions, via reductions of the gauge connection in Hermitian symmetric spaces. In such models the matter fields are coupled to gauge Chern-Simons fields, which are associated with the isotropy subgroup of the considered symmetric space. Moreover, they can be related to certain (integrable and non-integrable) evolution systems, as the Ishimori and the Heisenberg model. The main classical and quantum properties of these systems are discussed in connection with the topological field theory and the condensed matter physics.

[1] E-mail: martina@le.infn.it
[2] E-mail: pashaev@main1.jinr.dubna.su
[3] E-mail: soliani@le.infn.it
[4] Permanent address

I Introduction

In the last years, a new class of general covariant field theories, called quantum topological field theories (TFT's), were introduced [1, 2].These models were originally related to Yang-Mills instantons [1], σ-models [3] and gravity [4]. Successively, the work by Witten established a relationship between the covariant four-dimensional quantum fields and the abelian and non-abelian quantum Chern-Simons (CS) action on a three-dimensional manifold $\mathcal{M}$. The most relevant property of these theories is that of having observables (the Wilson line operators) which are metric independent. Moreover, in this framework, the vacuum expectation values of such observables are invariant under smooth deformations on $\mathcal{M}$, i.e. they are topological invariants of closed links in $\mathcal{M}$ and are related to the Jones polynomials in knot theory [5] and their generalizations.

From another point of view, models of point particles coupled to a CS gauge field in 2+1 dimensions, have gained attention owing to their peculiar long range interaction [6]. In particular, in the abelian CS theories the interacting point-particles (anyons) obey a fractional statistics [7, 8] and play a role in fractional quantum Hall effect and in high temperature superconductivity [9, 10].

Such a kind of theories can be extended to the non-abelian case [11] and can be treated as non-relativistic quantum systems [12, 13, 14, 15] . The quantization procedure leads to an N-body Schrödinger equation, with the Aharonov-Bohm potential in the abelian case, and with the presence of the Knizhnik-Zamolodchikov connection in the non-abelian case [11]. Moreover, in both cases, within the classical field approach equations of the gauged nonlinear Schrödinger (NLSE) type arise. In the static self-dual situation, such systems become the Liouville equation and its integrable multicomponent generalizations [12, 13, 14, 15, 16]. These results lead directly to a link between the static CS-theories and the static reductions of integrable equations in (2+1)- dimensions, like the Ishimori model and the Davey-Stewartson equation [17].

Now, the basic property of the integrable equations is that they can be represented as zero-curvature conditions for certain special linear connections (the Lax pair operators) in suitable spaces. Incidentally, this is a representation shared by the CS equations of motion in a three-manifold. Furthermore, among the great amount of consequences which derives from this fact,

a particular role is plaied by the transformation properties between two of such integrable equations. For example, it is well known that the NLSE is "gauge equivalent" to the Heisenberg model in 1+1 dimensions [18, 19, 20]. Furthermore, the method of the gauge equivalence has been extended to integrable (2+1)-dimensional equations, for instance between the Ishimori and the Davey-Stewartson equation (integrable extensions in (2+1)-dimensions of the Heisenberg model and of the NLSE, respectively) [21, 22, 23, 24]. However, this method can be made suitable for handling non-integrable systems also, at least under certain conditions. More specifically, the method is based on the interpretation of the spin variables as elements of a coset space $\mathcal{G}/\mathcal{H}$. Then, the geometric characterization of such a space is given in terms of the zero-curvature condition for the associated chiral currents.

In many physical applications [25], the coset space $\mathcal{G}/\mathcal{H}$ is taken to be a symmetric space, with isotropy group $\mathcal{H}$ [26, 27]. Hence, the current components, taking values in the Lie algebra of $\mathcal{H}$, are considered as the local gauge fields of the theory. The further degrees of freedom belongs to the tangent space of the symmetric space and are interpreted as the "matter fields". When this procedure is applied to a specific spin model, we refer to it as the "tangent space representation" of the given theory. In this formalism the spin dynamics provides further restrictions on the chiral currents, leading to some multicomponent gauged NLSE's. The integrable systems are recoverd only for a particular choice of the gauge connection.

In particular, by using this scheme we have mapped the continuous Heisenberg model in (2+1)-dimensions into an abelian pair of CS gauged NLSEs [28]. Thus, we have proved the existence of a connection between continuous ferromagnetic systems and CS models.

At this point, we notice that the classical field equations for the pure CS theory are formally just the zero-curvature conditions found in our tangent space approach to the planar ferromagnets. The only difference concerns the local gauge group, which is the whole $\mathcal{G}$ for the pure case, and the subgroup $\mathcal{H}$ for ferromagnets. Therefore, in the last case one has supplementary constraints, which are $\mathcal{H}$-invariant by construction, and break the general $\mathcal{G}$-invariance.

These considerations have suggested us to study the CS theory in the symmetric space $\mathcal{G}/\mathcal{H}$. Following this idea, in Section II we study such type of reductions and explain in which sense we obtain certain matter fields coupled to a local gauge CS field. Therefore, we show how one can introduce

suitable classical "gauge-fixing" conditions, which have to be invariant under gauge transformations on $\mathcal{H}$. Furthermore, the existence of a classification for the symmetric spaces [26, 27], and the requirement that the gauge fixing condition has to be invariant under specific subgroups, implies a classification of the possible theories obtained by using the outlined procedure. In Section III we study the generalized σ-models in the tangent space formalism. Thus, we show the deep relationship which exists between the spin models and the topological field theories. This relationship is studied in detail in Section IV, where the classical canonical structure of the $SU\left(2\right)/U\left(1\right)$ model is given. We show also how the constraints of the theory can be solved, and how the choice of the Heisenberg model as gauge-fixing condition yields a hamiltonian system, in which the $U\left(1\right)$ gauge-invariant and the non-invariant variables are separated. In Section V we discuss the quantization of the previous model and its connection with the multianyonic systems. Finally, Section VI is devoted to some remarks and open problems.

II Chern-Simons theory on symmetric spaces

The Chern-Simons action for a compact non-abelian simple Lie group $\mathcal{G}$ on an oriented closed three-dimensional manifold $\mathcal{M}$ is

$$S[J] = \frac{k}{4\pi}\int_{\mathcal{M}} Tr\ (J\wedge dJ + \frac{2}{3}J\wedge J\wedge J) \qquad \text{(II.1)}$$

where J is the 1-form gauge connection with values in the Lie algebra $\hat{g}$ of $\mathcal{G}$ and the trace is taken in a chosen representation. The action (II.1) is manifestly invariant under general coordinate transformations (preserving orientation and volumes). Moreover, under a generic gauge map $G:\mathcal{M}\ \rightarrow \mathcal{G}$ the gauge connection transforms as usual by $J\ \rightarrow\ G^{-1}JG + G^{-1}dG$. Correspondingly, the action (II.1) changes as $S[J]\ \rightarrow\ S[J] + 2\pi\, k\, W(G)$, where

$$W(J) = \frac{1}{24\pi^2}\int_{\mathcal{M}} Tr\ (G^{-1}dG\wedge G^{-1}dG\wedge G^{-1}dG) \qquad \text{(II.2)}$$

is the winding number of the map G and takes integer values, because of the result $\pi_3(\mathcal{G}\ ) = \mathbf{Z}$ of the homotopy theory [26] . The gauge invariance of the quantum theory (defined by the functional integral approach [1]-[4], or in the canonical formalism [29]) implies the quantization of the constant

k. Nevertheless, the infinitesimal gauge transformation $G = e^{\lambda} \simeq I + \lambda$ ($\lambda : \mathcal{M} \to \hat{g}$), acting on J as

$$\delta J = [J, \lambda] + d\lambda \qquad , \tag{II.3}$$

leaves the action invariant. The classical equations of motion for the action (II.1) is the zero-curvature condition

$$F = dJ + J \wedge J = 0 \qquad . \tag{II.4}$$

Now, let us choose in $\mathcal{G}$ a closed subgroup $\mathcal{H}$, such that the Lie algebra $\hat{g}$ of $\mathcal{G}$ satisfies the so-called $\mathbf{Z}_2$ - graduation condition

$$\hat{g} = \hat{l}^{(0)} \oplus \hat{l}^{(1)} \qquad , \qquad [\hat{l}^{(i)}, \hat{l}^{(j)}] \subset \hat{l}^{(i+j)} \,\mathrm{mod}\ (2) \qquad , \tag{II.5}$$

where $\hat{l}^{(0)}$ is the Lie algebra of $\mathcal{H}$ and $\hat{l}^{(1)}$ is the (vector space) complement of $\hat{l}^{(0)}$ in $\hat{g}$ [26, 27]. The group $\mathcal{G}$ acts transitively on the coset space $\mathcal{G}/\mathcal{H}$, which can be identified with a differential manifold belonging to the class of the symmetric spaces. The subgroup $\mathcal{H}$ leaving invariant the points of $\mathcal{G}/\mathcal{H}$ is called the isotropy group. At any point $p_0 \in \mathcal{G}/\mathcal{H}$, the tangent space $T_{p_0}\left(\mathcal{G}/\mathcal{H}\right)$ is isomorphic to $\hat{l}^{(1)}$ and a torsion-free connection can be canonically defined. Moreover, we can induce in a natural way a riemannian metric on $T\left(\mathcal{G}/\mathcal{H}\right)$, by restricting the Killing form on $\hat{g}$ to $\hat{l}^{(1)}$. However, for our aims we are interested in those symmetric spaces which enjoy a complex structure, the so-called Hermitian symmetric spaces [27], the main algebraic properties of which are:

1) there exists an element A belonging to the Cartan subalgebra of $\hat{g}$, such that its centralizer in $\hat{g}$ coincides with $\hat{l}^{(0)}$, thus $\left[A, \hat{l}^{(0)}\right] = 0$ holds as a particular case;

2) one can write $\hat{l}^{(1)} = \hat{l}^{(1+)} \oplus \hat{l}^{(1-)}$, where $\hat{l}^{(1\pm)} = \mathrm{span}_{\alpha\in\Phi^+}\left\{e_{\pm\alpha}\right\}$ are expressed by using those elements of the the Weyl-Cartan basis, which correspond to a subset Φ^+ of the positive root system on which $\alpha\left(A\right)$ is a constant, thus $\left[A, \hat{l}^{(1\pm)}\right] = \pm\alpha\left(A\right)\hat{l}^{(1\pm)}$ holds;

3) by using a proper scaling, $ad\,A$ can be considered as a linear involutive endomorphism on $\hat{l}^{(1)}$ supplying the complex structure on it ;

4) the commutation relations $[e_{\pm\alpha}, e_{\pm\beta}] = 0$ hold for all pairs of basis elements in $\hat{l}^{(1\pm)}$.

By using the graduation (II.5), we can first assume that the current J has the form

$$J = J^{(0)} + J^{(1)} \qquad , \tag{II.6}$$

where $J^{(i)}$ are 1-forms taking values in $\hat{l}^{(i)}$. Hence, by resorting to the properties 1) - 4), the CS action (II.1) becomes

$$S[J^{(0)}, J^{(1)}] = \frac{k}{4\pi} \int_{\mathcal{M}} Tr(J^{(0)} \wedge dJ^{(0)} + \frac{2}{3} J^{(0)} \wedge J^{(0)} \wedge J^{(0)} + J^{(1)} \wedge \hat{\mathrm{D}} J^{(1)}) \tag{II.7}$$

where $\hat{\mathrm{D}} = d + \cdot \wedge J^{(0)} + J^{(0)} \wedge \cdot$ is the covariant exterior derivative. The expression (II.7) suggests us to interprete $J^{(0)}$ as a CS-gauge field and $J^{(1)}$ as a coupled matter field. In this sense we have reformulated a $\mathcal{G}$ -invariant pure non-abelian CS theory (II.1) as an interacting matter gauge field theory with group $\mathcal{H}$. The whole set and the properties of such theories are directly connected with the classification problem of all possible Hermitian symmetric spaces, which is completely solved [27]. Moreover, although at first glance the distinction between matter and gauge fields could seem rather artificial, it is preserved under the action of $\mathcal{H}$. In fact, keeping in mind the infinitesimal transformation (II.3) and the $\mathbf{Z}_2$-graduation, with $\lambda = \lambda^{(0)} + \lambda^{(1)}$, we obtain

$$\begin{aligned} \delta J^{(0)} &= [J^{(0)}, \lambda^{(0)}] + d\lambda^{(0)} \\ \delta J^{(1)} &= [J^{(1)}, \lambda^{(0)}] \qquad . \end{aligned} \tag{II.8}$$

for $\lambda^{(1)} = 0$. Anyway, the action (II.7) still possesses the property of being $\mathcal{G}$ -invariant. Then, we have the freedom to introduce certain gauge fixing conditions, which are invariant under $\mathcal{H}$. This procedure could yield specific field models.

In order to display some concrete examples of such models, it is convenient to follow a suggestion by Witten [2]. This consists in "chopping" a general 3-manifold $\mathcal{M}$ into pieces, each of them is isomorphic to $\Sigma \times \mathbf{R}$, where Σ is a Riemann surface and $\mathbf{R}$ is interpreted as the time. In doing so, the exterior derivative d and the current J are expressed in terms of time and space components

$$d = d_0 + \mathbf{d} \qquad \left(d_0 = dx^0\, \partial_0\right) \tag{II.9}$$

and

$$J = A_0 + \mathbf{A} = A_0^{(0)} + A_0^{(1)} + \mathbf{A}^{(0)} + \mathbf{A}^{(1)} \qquad , \tag{II.10}$$

respectively. In the last equation we have taken into account the $\mathbf{Z}_2$-graduation (II.6) for J, $\mathbf{A}^{(i)} = A_a^{(i)}\, dx^a$ is a real connection on Σ and $A_0^{(i)} = A_0^{(i)}\, dx^0$ are 1-forms on $\mathbf{R}$. Furthermore, at this point we parametrize Σ by means of the local complex coordinates $z = x_1 + i\, x_2$, $\bar{z} = x_1 - i\, x_2$. Now, we can consider tensors, either of covariant or contravariant type, with some definite number of holomorphic and anti-holomorphic indeces. For instance, the cotangent space of Σ will be decomposed into the direct sum of two subspaces, one of them is spanned by dz, and the other by $d\bar{z}$. An important observation is that barred and unbarred tensors do not transform into each other under a holomorphic change of coordinates. Specifically, our connection J can be rewritten in the form

$$J = V_0 + V + \bar{V} + M_0 + M + \bar{M} \qquad , \tag{II.11}$$

where

$$V = \tfrac{1}{2}(A_1^{(0)} - iA_2^{(0)})\, dz\ , \qquad \bar{V} = \frac{1}{2}(A_1^{(0)} + iA_2^{(0)})\, d\bar{z} \qquad , \tag{II.12}$$

$$M = \tfrac{1}{2}(A_1^{(1)} - iA_2^{(1)})\, dz\ , \qquad \bar{M} = \frac{1}{2}(A_1^{(1)} + iA_2^{(1)})\, d\bar{z} \qquad , \tag{II.13}$$

and we put

$$V_0 = A_0^{(0)} \qquad , M_0 = A_0^{(1)} \tag{II.14}$$

for brevity. Correspondingly, we can split the exterior derivative in the form

$$\mathbf{d} = \partial + \bar{\partial} = dz\ \partial_z + d\bar{z}\ \partial_{\bar{z}} \qquad , \tag{II.15}$$

where ∂ e $\bar{\partial}$ are holomorphic and anti-holomorphic operators globally defined on Σ.

Taking into account this construction on $\Sigma \times \mathbf{R}$, the covariant exterior derivative introduced in the action (II.7) is written now as the sum of three terms: $D = \mathcal{D} + \bar{\mathcal{D}} + \mathcal{D}_0$, where

$$\mathcal{D} = \ \partial + V \wedge \cdot + \cdot \wedge V, \qquad \bar{\mathcal{D}} = \bar{\partial} + \bar{V} \wedge \cdot + \cdot \wedge \bar{V},$$
$$\mathcal{D}_0 = d_0 + V_0 \wedge \cdot + \cdot \wedge V_0. \tag{II.16}$$

Then, just by rearranging the $\mathcal{H}$ - invariant action (II.7) in the complex variables, we obtain the canonical Darboux form

$$S = \frac{k}{4\pi} \int_{\Sigma \times \mathbf{R}} \qquad Tr\ (V \wedge d_0\, \bar{V} + \bar{V} \wedge d_0\, V + M \wedge d_0\, \bar{M} + \bar{M} \wedge d_0\, M +$$

$$
\begin{aligned}
+2V_0 \wedge \Big(\partial\bar{V} + \bar{\partial}V + V\wedge\bar{V} + \bar{V}\wedge V + M\wedge\bar{M} + \bar{M}\wedge M\Big) \\
+2M_0 \wedge (\mathcal{D}\bar{M} + \bar{\mathcal{D}}M)) \qquad . \qquad\qquad (\mathrm{II}.17)
\end{aligned}
$$

The corresponding equations of motion are

$$
\begin{aligned}
\partial\bar{V} + \bar{\partial}V + V\wedge\bar{V} + \bar{V}\wedge V &= -\Big(M\wedge\bar{M} + \bar{M}\wedge M\Big)\,, && (\mathrm{II}.18)\\
d_0V + \partial V_0 + V\wedge V_0 + V_0\wedge V &= -\left(M_0\wedge M + M\wedge M_0\right), && (\mathrm{II}.19)\\
d_0\bar{V} + \bar{\partial}V_0 + \bar{V}\wedge V_0 + V_0\wedge\bar{V} &= -\Big(M_0\wedge\bar{M} + \bar{M}\wedge M_0\Big)\,, && (\mathrm{II}.20)\\
\mathcal{D}\bar{M} + \bar{\mathcal{D}}M &= 0 \quad , && (\mathrm{II}.21)\\
\mathcal{D}_0 M + \mathcal{D}M_0 &= 0 \quad , && (\mathrm{II}.22)\\
\mathcal{D}_0\bar{M} + \bar{\mathcal{D}}M_0 &= 0 \quad . && (\mathrm{II}.23)
\end{aligned}
$$

From the expression (II.17) the structure of the Lagrange density with contraints is manifest. In particular, V_0 and M_0 are the Lagrange multipliers enforcing the Gauss-Chern-Simons (GCS) law and a sort of generalized self-dual condition, which derives from the torsion - free property of the $\mathcal{G}\ /\mathcal{H}$ manifold. Furthermore, by using the infinitesimal transformations (II.8) and the expression (II.11), one easily sees that the action (II.17) and the corresponding equations of motion (II.18 - II.23) are explicitly gauge invariant under infinitesimal transformations defined on $\mathcal{H}$, in the sense that they are not only a subclass of the gauge symmetries admitted by the original model, but preserve the decomposition among gauge fields V-type and matter fields M-type. On the other hand, this picture induces us to figure out certain situations, in which the invariance under $\mathcal{G}$ -transformations is broken, in such a way that only the $\mathcal{H}$ -invariance is preserved. In other words, we may choose a supplementary constraint among the fields, which is invariant under the $\mathcal{H}$ - transformations. Since such a constraint have to transform accordingly to (II.8), its general form is

$$
\Gamma\left[M_0, M, \bar{M}, \mathcal{D}M_0, \bar{\mathcal{D}}M_0, \cdots\right] = 0 \qquad , \qquad\qquad (\mathrm{II}.24)
$$

where Γ denotes an arbitrary differentiable function, depending on the M-type fields and on their covariant derivatives. If Eq. (II.24) can be explicitly solved for M_0, then we replace this into the equations of motion (II.18 - II.23), obtaining nonlinear evolution equations for the matter fields M and

$\bar{M}$. The same substitution in the action (II.17) leads to a functional defined on the submanifold of $\mathcal{G}\ /\ \mathcal{H}$ determined by Eq. (II.21), which is seen as a further constraint. Conversely, in the case in which M_0 cannot be explicitly determined, we can include the constraint (II.24) into the action (II.17) by means of a suitable Lagrange multiplier. After all we have obtained special nonlinear evolution classical models for the matter fields in interaction with the CS field (generally non-abelian). Now, carrying out specific calculations in TFT one needs to break the gauge symmetry , usually by the Weyl gauge $A_0 = 0$ [2]. Thus the idea we want to stress is that other suitable constraints of the form (II.24) can be used, first because they may help in more effective calculations. In particular, when Eq. (II.24) is related to certain integrable systems, for which exact solutions can be given analitycally at least at the classical level. Secondly, they can provide integrable deformations of the topological symmetry of the original pure CS model. Furthermore, from Eqs. (II.18 - II.20) we obtain the identity

$$\begin{aligned}(\mathrm{dx}_0\partial_0 + [\mathrm{V}_0, \cdot]) \quad &\wedge \left(M \wedge \bar{M} + \bar{M} \wedge M\right) + (\partial + [V, \cdot]) \wedge \left(M_0 \wedge \bar{M} + \bar{M} \wedge M_0\right) \\ &+ \left(\bar{\partial} + \left[\bar{V}, \cdot\right]\right) \wedge (M_0 \wedge M - M \wedge M_0) = 0, \qquad \text{(II.25)}\end{aligned}$$

where we define the "commutator" operator over 1-forms $[A, \cdot] \wedge B = A \wedge B - B \wedge A$ yielding a 2-form. Eq. (II.25) yields a continuity equation, when a given model is defined by the specific dependence of M_0 on M and $\bar{M}$. The conserved density is given by the 2-form $\rho dz \wedge d\bar{z} = M \wedge \bar{M} + \bar{M} \wedge M$, which furnishes the set of non-abelian conserved charges $\mathcal{Q} = \int_\Sigma \rho\, dz \wedge d\bar{z}$.

A quite special situation occurs when we consider the Grassmann manifold $\mathcal{G}\ /\mathcal{H}\ = SU(n+m)/S(U(n) \times U(m))$ ($AIII$ in the Helgason's notation [27]). In particular, for $m = 1$ we obtain the complex projective $CP^n = SU\,(n+1)\,/S\,(U\,(n) \times U\,(1))$ model. In general, the gauge connection J is anti-hermitian ($J^\dagger = -J$) and decomposes in the block form

$$J = \begin{pmatrix} iV^{(n)} & R \\ Q & iV^{(m)} \end{pmatrix} \quad , \qquad \text{(II.26)}$$

where $V^{(l)} = v_\mu^{(l)} dx^\mu$ are $l \times l$ matrix valued one-forms with $l = n, m$, $Q = q_\mu dx^\mu$ and $R = r_\mu dx^\mu$ are $m \times n$ and $n \times m$ matrix valued one-forms, respectively. The anti-hermiticity of J implies that $v_\mu^{(l)\,\dagger} = v_\mu^{(l)}$ $(l = m, n)$

and $r_\mu = -q_\mu^\dagger$. Thus, in the complex formulation we introduce the forms

$$\begin{array}{llll} \psi_- dz = & \frac{1}{2}\left(q_1 - iq_2\right) dz & , \quad \psi_+ d\bar{z} = & \frac{1}{2}\left(q_1 + iq_2\right) d\bar{z} \quad , \\ v^{(n)} dz = & \frac{1}{2}\left({v^{(n)}}_1 - i{v^{(n)}}_2\right) dz & , \quad v^{(m)} dz = & \frac{1}{2}\left({v^{(m)}}_1 - i{v^{(m)}}_2\right) dz, \end{array} \tag{II.27}$$

which give a block decomposition of the matter and the gauge fields, M and V, respectively. Then, substitution into (II.17) yields the action in terms of $\psi_\pm$ and $v^{(i)}$. We shall not write this for brevity. Here we remark only that in the CP^n case the matter fields $\psi_\pm$ are n-component row vectors. Moreover, the cubic nonlinear self-interaction for the abelian $U(1)$ field vanishes identically in the action. This fact implies that the corresponding magnetic field $B^{(1)} = \epsilon^{ij}\,\partial_i\, v_j^{(1)}$ is proportional to the charge density $\rho^{(1)} = -i\left(\psi_+\psi_+^\dagger - \psi_-\psi_-^\dagger\right)$. Then the particles with electric charge $\mathcal{Q}_{el} = \int_\Sigma \rho\, dz \wedge d\bar{z}$ are also flux-tubes with total magnetic flux $\Phi_m = -\mathcal{Q}_{el}$. But for $n \geq 2$, accordingly to the equations of motion (II.18), the "coloured" magnetic field $B^{(n)} = \left[\partial_z \bar{v}^{(n)} - \partial_{\bar{z}} v^{(n)} + i\left(v^{(n)}\bar{v}^{(n)} - \bar{v}^{(n)} v^{(n)}\right)\right] = -i\left(\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-\right)$ appears. However, electric and "coloured" charges are not completely independent, since in general contributions from the non-abelian fields to the spatial components of the electric current are present (see Eq. (II.25)). Nevertheless, it is remarkable that the models of non-abelian CS field coupled to the matter [11] can be embedded into the scheme of our CP^n models, giving a specific example of constraint (II.24).

III Generalized σ-Models in the tangent space

In this Section we show how specific models can be embedded into the theory developed in Section II. In particular, we deal with generalized classical spin models, whose spin phase space is a symmetric space. There are several motivations for studying these type of systems. Many of them can be found in [25] and [30].

First of all we restrict ourselves to the 2 + 1 dimensional generalized Heisenberg model, which is a natural extension of 1+1 dimensional integrable version defined on symmetric spaces. As is well known from a long time [18, 19, 20], the latter model is gauge equivalent (in the sense of the completely integrable systems) to the NLSE. Thus, the NLSE describes the projection of the model on the tangent space of the spin phase space. Extending such a

representation to the 2+1 dimensional systems, surprisingly, a nonvanishing curvature connection and the Chern-Simons interaction appears [28].

Let us consider the matrix $\mathbf{S}$, which represents a point in the symmetric space $SU(n+m)/S\left(U\left(n\right)\times U\left(m\right)\right)$ and satisfies the constraint

$$\mathbf{S}^2 = \frac{1}{mn} I_{n+m} + \frac{(m-n)}{mn}\mathbf{S} \qquad , \tag{III.1}$$

where I_{n+m} stands for the identity matrix in $(n+m)$-dimensions. The corresponding Heisenberg model is defined by the equations of motion

$$i\partial_0\, \mathbf{S} = \frac{mn}{m+n}\left[\mathbf{S},\ \nabla^2\mathbf{S}\right] \tag{III.2}$$

This matrix can be diagonalized by a $U(n+m)$ local transformation g, namely

$$\mathbf{S} = g\Sigma g^{-1} \qquad , \Sigma = \begin{pmatrix} \frac{1}{n} I_n & 0 \\ 0 & -\frac{1}{m} I_m \end{pmatrix} \qquad . \tag{III.3}$$

Now, in order to construct the tangent space representation for this model we introduce the chiral current

$$J_\mu = g^{-1}\partial_\mu g = J_\mu^{(0)} + J_\mu^{(1)} \qquad (\mu = 0,1,2) \qquad , \tag{III.4}$$

where the $\mathbf{Z}_2$ graduation has been used. In particular, J_μ has the same block decomposition presented in Equation (II.26). Furthermore, by virtue of (III.4), the current J_μ can be considered as a gauge connection satisfying the zero-curvature condition, that is exactly the classical equation of motion (II.4) for the Chern-Simons topological field theory. Then, inspired by the discussion at the end of Section II, we can use the dynamical Eq. (III.2) in the tangent space formulation as a constraint for the current components, namely

$$i[J_0, \Sigma] = \frac{mn}{m+n}\left([\Sigma, [J_\mu [J_\mu, \Sigma]]] + [\Sigma, [\partial_\mu J_\mu, \Sigma]]\right) \qquad . \tag{III.5}$$

Thus, the dynamics for the currents J_μ is defined by the zero-curvature equation (II.4). From the decomposition (III.4) and by using the notation introduced in (II.26), we get

$$q_0 = i(\partial_j q_j + i v_j^{(m)} q_j - i q_j v_j^{(n)}) \qquad , \tag{III.6}$$

or, equivalently (see (II.27))

$$q_0 = 2i\left(D\,\psi_+ + \bar{D}\psi_-\right) \tag{III.7}$$

where we have introduced the "covariant" derivatives $D = \partial_z + iv^{(m)}\cdot - \cdot iv^{(n)}, \bar{D} = \partial_{\bar{z}} + iv^{(m)\dagger}\cdot - \cdot iv^{(n)\dagger}$. Equation (III.7) is an explicit example of the constraint (II.24). The substitution of q_0 in the action (II.17) gives the formulation of the generalized Heisenberg model (III.2) as a specific symmetric reduction from a pure non-abelian CS model

$$\begin{aligned}
S = \quad & \frac{k}{4\pi}\int_{\Sigma\times\mathbf{R}} Tr\ \{v^{(n)}\partial_0 v^{(n)\dagger} - v^{(n)\dagger}\partial_0 v^{(n)} + v^{(m)}\partial_0 v^{(m)\dagger} - v^{(m)\dagger}\partial_0 v^{(m)} \\
- \quad & 2v_0^{(n)}(\partial_z v^{(n)\dagger} - \partial_{\bar{z}} v^{(n)} + i[v^{(n)}, v^{(n)\dagger}]) \\
- \quad & 2v_0^{(m)}(\partial_z v^{(m)\dagger} - \partial_{\bar{z}} v^{(m)} + i[v^{(m}, v^{(m)\dagger}]) \\
+ \quad & \psi_+^\dagger D_0\psi_+ - \psi_+ D_0^\dagger\psi_+^\dagger - \psi_-^\dagger D_0\psi_- + \psi_- D_0^\dagger\psi_-^\dagger \\
+ \quad & 8i((D\psi_+)^\dagger D\psi_+ - (\bar{D}\psi_-)^\dagger\bar{D}\psi_-)\}\ dx^0 dz d\bar{z}
\end{aligned} \tag{III.8}$$

with the supplementary condition

$$\gamma \equiv D\psi_+ - \bar{D}\psi_- = 0 \qquad , \tag{III.9}$$

which is the Equation (II.21) and can be interpreted as the torsion-free condition for the spin phase space manifold.

In the action (III.8) we have used Tr as a global symbol for the trace in a given representation for $U(n)$ and $U(m)$, respectively, and $D_0 = \partial_0 + iv_0^{(m)}\cdot - \cdot iv_0^{(n)}$ in analogy with D and $\bar{D}$.

Let us notice here, that the tangent space representation is similar to the CP^1 formulation of the $O(3)$ nonlinear σ model [31]. But in this context the "matter fields" are identified with the complex coordinates of the group, while in our approach $\psi_\pm$ are given in terms of the first-order derivatives of the same quantities.

Now, we know that the classical Heisenberg model (III.2) is integrable for static configurations, thus we have relatively few information about the general properties of the solutions. From this point of view it is interesting to look at spin models which are integrable in (2+1)-dimensions. In such a case, the existence of a Lax pair enables one to linearize the time evolution of a generic initial configuration of the considered fields in terms of the corresponding spectral data, via the so-called Spectral Transform Method [32].

Thus, one may hope to obtain a great amount of information on the topological properties of the classical solutions. These observations suggest the existence of a deep relation between integrable systems and TFT. However, in (2+1)-dimensions the only known examples of integrable spin models are the Ishimori system [33] and the Manakov-Zakharov-Mikhailov-Ward chiral field equation [34, 35, 36]. The former system is related to the Davey-Stewartson equation [21, 22, 23, 24] and can be deduced as a special case of the so-called topological magnet model, which has been studied in [37, 38]. The equations of motion for the $SU(2)/U(1)$ topological magnet come from the Lax pair

$$\begin{aligned} L_1 &= \alpha I \partial_2 + \mathbf{S}\partial_1, \\ L_2 &= I\partial_0 + 2i\mathbf{S}\partial_1^2 + \left(i\partial_1 \mathbf{S} - i\alpha \mathbf{S}\partial_2 \mathbf{S} - \alpha \mathbf{S} w_2 + I w_1\right)\partial_1 \ , \end{aligned} \tag{III.10}$$

where $\mathbf{S} = \sum_{i=1}^3 S_i\, \sigma_i$ (the σ_i are the Pauli matrices) satisfies the relation $\mathbf{S}^2 = I_2$ and, furthermore, w_j is a $U(1)$ connection representing a velocity field, with $\alpha^2 = \pm 1$. Commuting these operators we obtain

$$i(\partial_0 + w^j \partial j)\mathbf{S} = \frac{1}{2}[\mathbf{S}, \partial^j \partial_j \mathbf{S}] + i\left(\partial^1 w_1 - \partial^2 w_2\right)\mathbf{S}, \tag{III.11}$$

$$\partial_i w_j - \partial_j w_i = -i\, Tr\left(\mathbf{S}[\partial_i \mathbf{S}, \partial_j \mathbf{S}]\right), \tag{III.12}$$

with the diagonal metric tensor $g^{ij} = (1, \alpha^2)$ on the flat space Σ. Equation (III.12) is a constraint on the vorticity of w_j and it is known, in the theory of the quantized vortices in the superfluid ^3He, as the Mermin-Ho relation [39]. If the velocity field w_j satisfies the incompressibility condition

$$\partial_1 w_1 + \alpha^2 \partial_2 w_2 = 0, \tag{III.13}$$

we obtain a ferromagnetic continuum model with non-trivial background, for which several vortex solutions were found and their dynamics described in [37, 40]. Furthermore, if w_j can be expressed in terms of one scalar function (the stream function) ϕ as

$$w_1 = \partial_2 \phi, \qquad w_2 = \alpha^2 \partial_1 \phi \tag{III.14}$$

we obtain the Ishimori model

$$\begin{aligned} i(\partial_0 \quad &+ \quad \partial_1 \phi \partial_2 + \partial_2 \phi \partial_1)\mathbf{S} = \frac{1}{2}[\mathbf{S}, (\partial_1^2 + \alpha^2 \partial_2^2)\mathbf{S}] \qquad , \\ (\quad \partial_1^2 \quad &-\alpha^2 \partial_2^2)\phi = -i\alpha^2 \mathbf{S}[\partial_1 \mathbf{S}, \partial_2 \mathbf{S}] \qquad . \end{aligned} \tag{III.15}$$

In the tangent space representation, this system takes the form

$$q_0 = i(D^j + iw^j)q_j, \tag{III.16}$$

$$\left(\partial_1^2 - \alpha^2\partial_2^2\right)\phi = -4i(\bar{q}_1 q_2 - \bar{q}_2 q_1)\ , \tag{III.17}$$

where w is defined by (III.14). Solving Eq. (III.17) for ϕ under suitable boundary conditions, and replacing into (III.16), we obtain a nonlocal constraint for q_0, analogous to (III.7). In this way we realize a new type of gauge fixing condition (II.24), which is integrable.

Finally, in the context of Heisenberg models we mention the possibility to choose a different constraint of the type (II.24), which cannot be solved explicitly for q_0. Indeed, if we consider the generalized relativistic model for the antiferromagnetic ground state [30], in the tangent space representation it reads

$$\partial_\mu q_\mu + \ \mathrm{iV}_\mu^{(m)} q_\mu - \ \mathrm{iq}_\mu V_\mu^{(n)} = 0 \qquad (\mu = 0, 1, 2) \qquad . \tag{III.18}$$

Because of the complete space-time symmetric form of these equations, we can add them as constraints to the action (II.17) by introducing a suitable Lagrange multiplier. Such constraints can be interpreted as a non-abelian Lorentz gauge condition for the TFT.

IV Canonical structure of the $SU(2)/U(1)$ model

In this Section, after some general considerations concerning the Hamiltonian structure of the theories (II.1) - (II.17) , we perform a more detailed treatment of the $SU(2)/U(1)$ case.

Resorting to the "trivialization" of the three manifold $\mathcal{M} \equiv \Sigma \times \mathbf{R}$ (see Sec. 2) and following the standard prescriptions [2], we equip the general non-abelian CS model (II.1) with the canonical structure given by

$$\left\{A_i^a(\mathbf{x}), A_j^b(\mathbf{y})\right\} = \frac{4\pi}{k}\varepsilon_{ij}\delta^{ab}\delta(\mathbf{x}-\mathbf{y}) \qquad , \tag{IV.1}$$

where A_i^a $(i = 1, 2)$ are the spatial components of the connection J (see Eq. (II.10)) represented in a certain basis of the Lie algebra $\hat{g}$. Now, it is natural

to use the $\mathbf{Z_2}$-graduation (II.5) for these canonical variables, expressed in terms of the fields $V = V\,dz$, $\bar{V} = \bar{V}\,d\bar{z}$, $M = M\,dz$ and $\bar{M} = \bar{M}\,d\bar{z}$ introduced in (II.11). Thus, the nonvanishing Poisson brackets are

$$\left\{ V^a(\mathbf{x}), \bar{V}^b(\mathbf{y}) \right\} = \frac{2\pi i}{k}\delta^{ab}\delta(\mathbf{x}-\mathbf{y}) \qquad ,$$
$$\left\{ M^\alpha(\mathbf{x}), \bar{M}^\beta(\mathbf{y}) \right\} = \frac{2\pi i}{k}\delta^{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}) \qquad , \tag{IV.2}$$

where V_i^a, $\bar{V}_i^b$ span the subalgebra $\hat{l}^{(0)}$. Analogously, M^α, $\bar{M}^\beta$ span the subalgebra $\hat{l}^{(1)}$. This structure supplies a further motivation for the interpretation of M and $\bar{M}$ as matter fields interacting with the non-abelian CS gauge fields V, $\bar{V}$.

The simplest case pertinent to such theories with abelian gauge field is the $SU(2)/U(1)$ model, whose action is straigthforwardly derived from (II.17) and (II.26) - (II.27):

$$S = -\frac{k}{\pi}\int_{\Sigma\times\mathbf{R}} \left\{ \frac{1}{2}\epsilon^{\lambda\mu\nu}\, v_\lambda \partial_\mu v_\nu \right.$$
$$+ i\frac{1}{2}\left(\psi_+^* D_0\psi_+ - \psi_+ (D_0\psi_+)^* - \psi_-^* D_0\psi_- + \psi_- (D_0\psi_-)^* \right) \tag{IV.3}$$
$$\left. - iq_0^*\left(D\psi_+ - \bar{D}\psi_- \right) + iq_0\left(D\psi_+ - \bar{D}\psi_- \right)^* \right\} dx^0\, dx^1\, dx^2 \qquad ,$$

where $D_0 = \partial_0 - 2iv_0$, $D = \partial_z - 2iv$ and $\bar{D} = \partial_{\bar{z}} - 2iv^*$ with $v = \frac{1}{2}(v_1 - iv_2)$. From (IV.2) we obtain the corresponding canonical Poisson structure related to (IV.3), namely

$$\{ v_i(\mathbf{x}), v_j(\mathbf{y}) \} = -\frac{\pi}{k}\varepsilon_{ij}\delta(\mathbf{x}-\mathbf{y}) \;, \left\{ \psi_\pm(\mathbf{x}), \bar{\psi}_\pm(\mathbf{y}) \right\} = \pm\frac{\pi i}{k}\delta(\mathbf{x}-\mathbf{y}) \tag{IV.4}$$

In analogy with our observation about the action (II.17), in (IV.3) we are considering v_0 and q_0 as Lagrange multipliers, enforcing the constraints of the model, i.e. the GCS law $\Gamma_1 = \partial_1 v_2 - \partial_2 v_1 + 2\left(|\psi_+|^2 - |\psi_-|^2\right)$, and the complex "torsion-free" constraint $\gamma = D\psi_+ - \bar{D}\psi_-$, which specializes Eq. (III.9) to the abelian case. They generate the $su(2)$ algebra of the gauge symmetry transformations. Furthermore, the GCS law constraint Γ_1 generates the $U(1)$ subgroup of local gauge transformations. Moreover, following the Dirac's classification of the constraints, we can introduce the set of primary constraints $\Gamma_0 \equiv \pi_0 = 0$, $\Gamma_2 \equiv \pi_q = 0$, where π_0 is the momentum

conjugated to the canonical variable v_0 and π_q is the momentum conjugated to q_0 such that

$$\{v_0, \pi_0\} = -\frac{\pi}{k}\delta(\mathbf{x}-\mathbf{y}) \qquad , \{q_0, \pi_q\} = \frac{\pi i}{k}\delta(\mathbf{x}-\mathbf{y}) \qquad . \tag{IV.5}$$

Hence, we can write the Hamiltonian $H = \int_\Sigma \mathcal{H}\ d^2x$ in terms of a set of first-class constraints only [41]. Indeed, we have

$$\mathcal{H} = \frac{k}{\pi}\left(v_0\Gamma_1 + iq_0\gamma^* - iq_0^*\gamma - f_0\Gamma_0 + ig_0\Gamma_2 - ig_0^*\Gamma_2^*\right) \qquad , \tag{IV.6}$$

where now the GCS law $\Gamma_1 = \{\pi_0, H\}$, $\gamma = \left\{\pi_q^*, H\right\}$ and $\gamma^* = \{\pi_q, H\}$ are to be considered now as secondary constraints. We see also that all constraints $\{\Gamma_1, \gamma\} \otimes \{\Gamma_0, \Gamma_2\}$ form a closed Lie algebra with abelian radical, and their dynamics is also weakly invariant ($\{\gamma, H\} \approx \{\Gamma_i, H\} \approx 0$ for $i = 0, 1, 2$). Furthermore, one easily checks that the determinant of their Poisson brackets is vanishing. The Hamiltonian itself is weakly vanishing, since it is a linear combination of constraints. Finally, in the expression (IV.6) f_0, g_0 and g_0^* are arbitrary functions, which characterize the evolution of v_0, q_0 and q_0^*, respectively.

At this stage we recall that, following the Dirac ideas [41], in [45] it is suggested to formulate both classical and quantum $U(1)$ CS theories in 2+1 dimensions in a completely invariant way. This result is achieved by introducing in the hamiltonian formalism new momenta, which are equal to the constraints imposed by the gauge theory. In this case, the corresponding new conjugated coordinates are pure gauge functions. Thus, one can provide a Hamiltonian depending only on the gauge invariant degrees of freedom. Here we show that this separation between two distinguished classes of variables cannot be performed up to the end, at least when only a gauge subgroup is explicitly solved. To be precise, we solve the GCS constraint, associated with the $U(1)$ symmetry, and reformulate the Hamiltonian (IV.6) in terms of the $U(1)$-invariant degrees of freedom and of pure gauge coordinates. However, we cannot separate the Hamiltonian in two pieces, one of which contains only $U(1)$-invariant canonical variables. This phenomenon seems to be connected with the non-abelian structure of the algebra of constraints in a completely constrained Hamiltonian. Our point of view is that this algebra cannot be "strongly abelianized" by using a unique regular canonical transformation.

Following the scheme of [45], first we restrict ourselves to the planar geometry taking $\Sigma \equiv \mathbf{R}^2$ and the rotational covariance of the theory, by expressing v_i into a longitudinal and a transverse part

$$v_i(\mathbf{x}) = \partial_i\, \eta(\mathbf{x}) - \epsilon_{ij}(\partial_j^{-1}B)(\mathbf{x}) \qquad , \tag{IV.7}$$

where we have applied the operator $\partial_j^{-1} f\,(\mathbf{x}) = \frac{1}{2\pi}\partial_j^{(x)} \int \ln|\mathbf{x}-\mathbf{y}| f(\mathbf{y}) d^2y$ to the magnetic field $B = \epsilon^{ij}\partial_i v_j$ (notice that $\partial_1\partial_1^{-1} + \partial_2\partial_2^{-1} = 1$). Analogously, it is convenient to express $\psi_\pm$ in terms of the canonical variables $(Q_\pm, P_\pm)$ by

$$\psi_\pm = \sqrt{\frac{\pi}{2k}}\,(Q_\pm \mp iP_\pm)\,. \tag{IV.8}$$

Then, the Poisson structure is given by

$$\{Q_\pm(\mathbf{x}), P_\pm(\mathbf{y})\} \;=\; \left\{\eta(\mathbf{x}), \frac{k}{\pi}B(\mathbf{y})\right\} = \delta(\mathbf{x}-\mathbf{y}), \tag{IV.9}$$

$$\{v_0(\mathbf{x}), \pi_0(\mathbf{y})\} \;=\; \{q_0(\mathbf{x}), \pi_q(\mathbf{y})\} = \delta(\mathbf{x}-\mathbf{y}). \tag{IV.10}$$

Any other Poisson bracket vanishes. Equations (IV.9) and (IV.10) have been derived using the rescaled fields $-\frac{k}{\pi}\, v_0 \to v_0$, $\sqrt{\frac{k}{\pi}}\, q_0 \to q_0$ and $-i\sqrt{\frac{k}{\pi}}\, \pi_q \to \pi_q$.

Now, we look for a suitable canonical transformation, such that some of the new momenta are equal to the constraints. In such a way, the coordinates canonically conjugated to these momenta have an arbitrary time evolution and are remnants of the gauge invariance of the theory.

First, let us denote by $\left(\tilde{Q}_\pm, \tilde{P}_\pm\right)$, $(\tilde{v}_0, \tilde{\pi}_0)$, $(\tilde{\eta}, \tilde{\pi}_1)$, $(\tilde{q}_0, \tilde{\pi}_2)$ new canonically conjugated coordinates, where the momenta $\tilde{\pi}_i$ have to be set equal to the constraints Γ_i, $\quad (i = 0, 1, 2)$. This can be done by means of a suitable generating function

$$W = W\left(Q_\pm, v_0, \eta, q_0; \tilde{P}_\pm, \tilde{\pi}_0, \tilde{\pi}_1, \tilde{\pi}_2\right) \qquad , \tag{IV.11}$$

satisfying the set of generalized Hamilton-Jacobi equations

$$\tilde{\pi}_i = \Gamma_i\left(Q_\pm, v_0, \eta, q_0; \frac{\partial W}{\partial Q_\pm}, \frac{\partial W}{\partial v_0}, \frac{\partial W}{\partial \eta}, \frac{\partial W}{\partial q_0}\right). \tag{IV.12}$$

The integrability of this system is assured by the commutation property $\{\Gamma_i, \Gamma_j\} = 0$. In fact, its general solution is given by

$$W = \tilde{\pi}_0 v_0 + \tilde{\pi}_2 q_0 + \tilde{\pi}_2^* q_0^* + \frac{k}{\pi}\,\left(\tilde{\pi}_1 + \frac{k}{\pi}\tilde{P}_-^2 - \frac{k}{\pi}\,\tilde{P}_+^2\right)\eta$$

$$+\epsilon_+ \int^{Q_+} \sqrt{\tilde{P}_+^2 - Q_+^2}\, dQ_+ - \epsilon_- \int^{Q_-} \sqrt{\tilde{P}_-^2 - Q_-^2}\, dQ_- \qquad \left(\epsilon_\pm^2 = 1\right). \tag{IV.13}$$

The corresponding canonical transformation reads

$$\begin{aligned} \pi_0 &= \tilde{\pi}_0,\ v_0 = \tilde{v}_0,\ \pi_2 = \tilde{\pi}_2,\ \pi_2^* = \tilde{\pi}_2^*,\ q_0 = \tilde{q}_0,\ q_0^* = \tilde{q}_0^*, & \text{(IV.14)} \\ B &= \tilde{\pi}_1 + \frac{\pi}{k}\left(\tilde{P}_-^2 - \tilde{P}_+^2\right), \qquad \eta = \frac{\pi}{k}\,\tilde{\eta}, & \text{(IV.15)} \end{aligned}$$

$$\begin{aligned} Q_\pm &= \epsilon_\pm \left|\tilde{P}_\pm\right| \sin\left(\frac{\tilde{Q}_\pm}{\tilde{P}_\pm} \pm \frac{2\pi}{k}\tilde{\eta}\right), \\ P_\pm &= \epsilon_\pm \left|\tilde{P}_\pm\right| \cos\left(\frac{\tilde{Q}_\pm}{\tilde{P}_\pm} \pm \frac{2\pi}{k}\tilde{\eta}\right). \end{aligned} \tag{IV.16}$$

Now, we define the new gauge-invariant degrees of freedom

$$\Phi_\pm = \pm \frac{i\epsilon_\pm}{\sqrt{2}} \left|\tilde{P}_\pm\right| \exp\left(\pm i\, \frac{\tilde{Q}_\pm}{\tilde{P}_\pm}\right) = \sqrt{\frac{k}{\pi}}\ \psi_\pm \exp\left(-2i\eta\right), \tag{IV.17}$$

which fulfill the canonical brackets

$$\{\Phi_\pm(\mathbf{x}), \bar{\Phi}_\pm(\mathbf{y})\} = \pm\, i\, \delta(\mathbf{x} - \mathbf{y}). \tag{IV.18}$$

In terms of these new variables, the Hamiltonian density becomes

$$\mathcal{H} = -v_0\pi_1 + iq_0\gamma^* - iq_0^*\gamma - f_0\pi_0 - g_0\pi_2 - g_0^*\pi_2^*, \tag{IV.19}$$

where we have dropped the "~" for simplicity. Furthermore, the constraint γ takes the form

$$\gamma \equiv \exp\left(\frac{2\pi i}{k}\eta\right)\left[\left(\partial_z - \frac{1}{2}\partial_{\bar{z}}^{-1}(B)\right)\Phi_+ - \left(\partial_{\bar{z}} - \frac{1}{2}\partial_z^{-1}(B)\right)\Phi_-\right] = 0, \tag{IV.20}$$

where the operator $\partial_{\bar{z}}^{-1}\left(f\left(z, \bar{z}\right)\right) = i\partial_z \int \frac{1}{\pi} ln\left|z - \xi\right| f\left(\xi, \bar{\xi}\right)\, d\xi \wedge d\bar{\xi}$ acts on $B = \pi_1 + \frac{\pi}{k}\left(\left|\Phi_-\right|^2 - \left|\Phi_+\right|^2\right)$. From this expression we see that γ still contains explicitly the gauge variables (η, π_1) through the exponential factor and the expression of B given above. However, we cannot extract them from γ in order to get a completely $U(1)$-gauge invariant constraint without the introduction of second class constraints. Moreover, the equations of motion

for $\Phi_\pm$, deriving from the Hamiltonian (IV.19), still involve q_0, which has an arbitrary dynamics. This happens because $\Phi_\pm$ are $U(1)$-gauge invariant only.

Moreover, the classical dynamics is restricted on a manifold of solutions defined by the constraints

$$\pi_1 = 0, \qquad \gamma = 0 \quad , \tag{IV.21}$$

which become

$$B = \frac{2\pi}{k}\left(|\Phi_-|^2 - |\Phi_+|^2\right) \tag{IV.22}$$

$$\left(\partial_z - \frac{1}{2}\partial_{\bar z}^{-1}(B)\right)\Phi_+ \quad - \left(\partial_{\bar z} + \tfrac{1}{2}\partial_z^{-1}(B)\right) \quad \Phi_- = 0 \quad . \tag{IV.23}$$

These equations enable us to look for time independent solutions, since no evolution operator is included. On the other hand, from the general theory of the completely constrained systems [51], we know that all dynamical informations are encoded into general canonical transformations, which can be considered as gauge transformations and time reparametrizations. In such a way we can generate time-dependent solutions from the static ones.

A special subcase of Eqs. (IV.23), corresponding to the self-dual CS model, is obtained when Φ_- vanishes (or, in alternative, Φ_+). In fact, by setting $\Phi_+ = \rho^{\frac{1}{2}}\, e^{i\chi}$ it easy to see that the previous equations reduce to the Liouville equation

$$\nabla^2 \ln\, \rho = -\frac{8\pi}{k}\rho, \tag{IV.24}$$

whose general solution is given in terms of an arbitrary holomorphic function [50] ζ by the relation

$$\rho = \frac{k}{4\pi}\,\frac{|\partial_z\zeta|^2}{\left(1+|\zeta|^2\right)^2} \quad . \tag{IV.25}$$

The phase χ is an arbitrary harmonic function. In particular, choosing $\zeta = \sum_{n=1}^{N}\frac{c_n}{z-z_n}$, where c_n and z_n are arbitrary complex numbers, and assuming that the phase χ is a regular and singlevalued function, we obtain the so-called self-dual CS solitons [14]. Finally, let us observe that since the Liouville equation is conformally invariant, all reductions we have performed on the original CS model preserve this special infinite dimensional symmetry group.

A more general situation occurs when (see [28])

$$\left(\partial_z - \frac{1}{2}\partial_{\bar{z}}^{-1}(B)\right)\Phi_+ = \left(\partial_{\bar{z}} + \frac{1}{2}\partial_z^{-1}(B)\right)\Phi_- = 0 \qquad , \tag{IV.26}$$

where both the fields $\Phi_\pm$ are different from zero. Let us notice that the system of these equations with Eq. (IV.22) corresponds to the so-called Hitchin equation introduced as a reduction of the self-dual Yang-Mills equations in 4 dimensions [52].

The relations (IV.26) suggest to introduce a holomorphic function U ($\partial_{\bar{z}}U = 0$) defined by

$$U(z,\bar{z}) = \bar{\Phi}_+(z,\bar{z})\ \Phi_-(z,\bar{z}) . \tag{IV.27}$$

This quantity is the analogous of the holomorphic component of the energy-momentum stress tensor in the Conformal Field Theories [53]. Now, let us suppose that U is a given entire function. Then, we can solve for instance (IV.27) with respect to Φ_- and write down the equations

$$\begin{aligned} \nabla^2\sigma &= -\frac{16\pi}{k}\ \mathrm{e}^{\eta}\sinh\sigma \\ \nabla^2\eta &= \nabla^2\chi = 0 \qquad , \end{aligned} \tag{IV.28}$$

where we have introduced $\sigma = \ln\left(|\Phi_+|^2 \,/\, |U|\right)$, $\eta =\ \mathrm{Re}\ln U$ and $\chi = \arg(\Phi_+)$. Equations (IV.28) are conformally invariant and are strictly related to the integrable conformal invariant affine Toda field theory (see [54]). Actually, this model contains three fields. Two of them correspond to σ and η in Eq. (IV.28). The third one is completely determined in terms of the former. Now, some comments are in order. Precisely, in the limit $\eta \to 0$, Eq. (IV.28) becomes the so-called sinh-Gordon equation. Conversely, in the limit $\eta \to \infty$ we recover the Liouville equation. All these models are completely integrable systems and their solutions can be studied by resorting to the Inverse Spectral Transform (IST) method. In particular, for $k > 0$ the sinh-Gordon equation admits solutions with pointlike singularities, whose behavior at large distances is given by $\sigma \to A\, K_0(r)$, where K_0 is the modified Bessel function of the second kind of order 0 [55].However, such a solution cannot be given in closed form and is related to the Painlevé transcendents. Multicharged solutions have been also dicussed [56].

For $k < 0$ one can obtain analytic solutions satisfying the constant boundary conditions on a finite rectangular boundary. Thus, by using the IST method for finite gap solutions, one can find multiperiodic solutions in terms of the Riemann θ function [57]. On the other hand, concerning the sinh-Gordon equation, many possible interpretations exist, which are different from the affine conformal Toda model mentioned above. For instance, it can be regarded as a perturbation of the free massless conformal model, or as a perturbation of the conformal Liouville model [53]. It was studied in connection with a two-component Coulomb gas [58] or an anyon-anti-anyon system [28]. In a different context it was used in the study of a system of vortices in a bounded magnetized plasma [59] and of the counter-rotating vortices in an inviscid and incompressible fluid [60].

Now, we can figure out that among all possible evolutions associated with the model (IV.3) we can select some of them which seem to be particularly interesting from the point of view of their physical interpretation. Indeed, instead of using the Weyl condition which in our case reads $v_0 = 0$ and $q_0 = 0$, we can choose the multiplier q_0 to be precisely that prescribed by Eq. (III.7). This means: i) that the action will contain only the fields $\psi_\pm$ and v_μ, ii) in order to recover the topological symmetry the constraint γ must be added to the equations of motion obtained by the new action. The system arising in this way is the $SU(2)/U(1)$ Heisenberg model in the tangent space representation, that is the abelian reduction of model (III.8). It contains the difference of two Pauli actions for non-relativistic charged scalar matter fields ψ_+ and ψ_-, coupled to an abelian CS field v_μ.

At this point, first we notice that the mapping between Heisenberg model and CS model, provided by Eqs. (II.26-II.27) and (III.3-III.4), enables us to express in a natural way the magnetic field in terms of the charge density (see Section II). Moreover, the magnetic energy of the Heisenberg system is interpreted directly in terms of the "number of particles" of the CS gauged model. In fact, after some simple algebraic manipulations, the equations of motion deriving from expression (III.8) reduce to a pair of NLSE for $\psi_\pm$ coupled through the abelian CS gauge field and constrained by Eq. (III.9) (see [28]). This situation is quite similar to that of Ref. [13, 14, 15]. In the static case, i.e. in the "quasi-Weyl" condition $q_0 = 0$, these equations reduce to the self-dual configurations discussed above. In this context, classical self-dual CS solitons and multiperiodic solutions can be interpreted in terms of magnetic bubbles (vortices) in the spin planar model [17]. These configurations have

spin angular momentum proportional to $\mathcal{Q}_{el}^2$ [42]. The vortices possess topological properties, which are invariant under general gauge transformations like (II.3). Then, we can speculate that vortex solutions i) can be found in the other spin models obtained by a different gauge choice, ii) they describe a special sector in the moduli space of the original TFT (II.1).

Moreover, we found the Hamiltonian structure of the Heisenberg model in the tangent space representation, by resorting to the symplectic method of Hamiltonian reduction [44] addressed to the gauge-invariant approach [45] which leads, in this case, to a separation of the gauge variables from the gauge-invariant Hamiltonian [43, 42]. In terms of the $U(1)$-gauge invariant degrees of freedom $\Phi_\pm$, the Hamiltonian density is given by

$$\mathcal{H} = 4\left|\left(\partial_{\bar{z}} + \frac{1}{2}\partial_z^{-1}(B)\right)\Phi_-\right|^2 - 4\left|\left(\partial_z - \frac{1}{2}\partial_{\bar{z}}^{-1}(B)\right)\Phi_+\right|^2 - f_0\pi_0 - v_0\pi_1, \tag{IV.29}$$

where we do not need to introduce the conjugated momenta to q_0, q_0^* and the related primary constraints. Since the last two terms in (IV.29) have vanishing Poisson brackets with $\Phi_\pm$, they can be set strongly equal to zero, when we study the time evolution of the gauge-invariant degrees of freedom given by $\dot{\Phi}_\pm = \{\Phi_\pm, H\}$.

By this procedure we have selected a particular time evolution, among all reparametrization transforms admitted by the pure $SU(2)$ model (IV.3). But we could repeat the same argument using a different choice of q_0, for instance the expression (III.16) for the Ishimori model. The integrability of this model will provide a complete description of the corresponding phase space. Thus we expect that this could improve the study of the original TFT.

V Quantization of the $SU(2)/U(1)$ model

The quantum theory of the model (IV.3) can be carried out by means of the correspondence $\Phi_\pm \to \hat{\Phi}_\pm$ and $\bar{\Phi}_\pm \to \hat{\Phi}_\pm^\dagger$ and replacing the canonical brackets by the equal-time commutators in the boson case (or the anticommutators in the fermionic case)

$$[\hat{\Phi}_\pm(\mathbf{x}), \hat{\Phi}_\pm^\dagger(\mathbf{y})] = \mp\,\delta(\mathbf{x}-\mathbf{y}). \tag{V.1}$$

The presence of a different signature in Eq. (V.1) leads generally to an unbounded quantum energy spectrum, whose treatment requires some special care. However, we know that the classical value of the energy is zero because of the completely constrained character of the classical Hamiltonian. So we expect that all the physical quantum states have to be eigenstates corresponding to the energy eigenvalue 0. Furthermore, the first class constraints Γ_i become the operators $\hat{\Gamma}_i$, which must annihilate the physical states. In particular, the operator $\hat{\Gamma}_1$ associated with the GCS law has to annihilate the physical states. This implies that such states are independent from v_0 and are invariant under time-independent gauge transformations. Therefore, all the operators $\hat{\Gamma}_i$ must commute among themselves. Finally, we have to associate with the constraint γ a corresponding operator $\hat{\gamma}$, which also annihilates the physical states.

For the specific case of the Heisenberg model, in the subspace of the physical states, we can write down a quantum Hamiltonian involving only the operators $\hat{\Phi}_\pm$ and their hermitians:

$$\hat{H} = 4 \int \left\{ \hat{\Phi}_+^\dagger \left(\partial_z - \frac{1}{2} \partial_{\bar{z}}^{-1} \left(\hat{B} \right) \right)^2 \hat{\Phi}_+ - \hat{\Phi}_-^\dagger \left(\partial_{\bar{z}} - \frac{1}{2} \partial_z^{-1} \left(\hat{B} \right) \right)^2 \hat{\Phi}_- \right\} dz d\bar{z}, \tag{V.2}$$

where $\hat{B} = \frac{\pi}{k} \left(\hat{\Phi}_-^\dagger \hat{\Phi}_- - \hat{\Phi}_+^\dagger \hat{\Phi}_+ \right)$, and the normal ordering of the operators is used. The quantized free-torsion constraint $\hat{\gamma}$ takes the form

$$\hat{\gamma} = \left(\partial_z - \frac{1}{2} \partial_{\bar{z}}^{-1} \left(\hat{B} \right) \right) \hat{\Phi}_+ - \left(\partial_{\bar{z}} - \frac{1}{2} \partial_z^{-1} \left(\hat{B} \right) \right) \hat{\Phi}_- \quad . \tag{V.3}$$

Therefore, since $\left[\hat{B} \left(\mathbf{x} \right), \hat{\eta} \left(\mathbf{y} \right) \right] = -\frac{\pi i}{k} \delta(\mathbf{x} - \mathbf{y})$ holds, one has the relation

$$\exp \left(i \, \hat{\eta} \left(\mathbf{y} \right) \right) \hat{B} \left(\mathbf{x} \right) \exp \left(-i \, \hat{\eta} \left(\mathbf{y} \right) \right) = \hat{B} \left(\mathbf{x} \right) - \frac{\pi}{k} \delta^2 \left(\mathbf{x} - \mathbf{y} \right). \tag{V.4}$$

This result is exploited to prove that $\hat{\Phi}_\pm$ are the $U(1)$-gauge invariant operators, which create a charge-solenoid composite, having magnetic flux equal to $\mp \pi / k$.

Now, we define the quantum vacuum state by the relation $\hat{\Phi}_\pm |\mathbf{0}>= 0$. Thus we can introduce two different particles number operators

$$\hat{N}_\pm = \int \hat{\Phi}_\pm^\dagger \hat{\Phi}_\pm \, d^2 x \quad . \tag{V.5}$$

These operators commute between themselves and with the Hamiltonian operator (V.2). Thus, we can formally construct the common eigenstates of the energy and of the occupation numbers for both types of particles by

$$\begin{aligned} |N_+, N_- > \ &= \int \prod_{i=1}^{N_+} d^2x_i^+ \prod_{j=1}^{N_-} d^2x_j^- \, \Psi\left(\mathbf{x}_1^+, \ldots, \mathbf{x}_{N_+}^+, \mathbf{x}_1^-, \ldots, \mathbf{x}_{N_-}^-\right) \\ & \hat{\Phi}_+^\dagger\left(\mathbf{x}_1^+\right) \ldots \hat{\Phi}_+^\dagger\left(\mathbf{x}_{N_+}^+\right) \hat{\Phi}_-^\dagger\left(\mathbf{x}_1^-\right) \ldots \hat{\Phi}_-^\dagger\left(\mathbf{x}_{N_-}^-\right) |\mathbf{0}> . \end{aligned} \qquad (V.6)$$

The function Ψ is an arbitrary element of the Hilbert space $L_2[\mathcal{R}^{2(N_+ + N_-)}]$ obeying the Schrödinger equation for $(N_+ + N_-)$-bodies.

Now, to be physical, the state (V.6) has to be also annihilated by $\hat{\gamma}$. For states with finite number of particle we can find exactly solvable equations for Ψ only when one type of particles is present. For instance, the $|N_+, 0\rangle$ state is described by the bosonic wave function

$$\Psi\left(\mathbf{x}_1^+, \cdots, \mathbf{x}_{\mathrm{N}_+}^+\right) = \mathcal{F}\left(\overline{z}_1^{\,+}, \cdots, \overline{z}_{N_+}^{\,+}\right) \prod_{i<j} \left|\mathbf{x}_{\mathrm{i}}^+ - \mathbf{x}_{\mathrm{j}}^+\right|^{-\frac{2}{k}} \quad , \qquad (V.7)$$

where $\mathcal{F}$ is an arbitrary holomorphic function of its arguments. By using a singular gauge transformation [46, 47], the expression (V.7) takes the form of the Laughlin multivalued anyonic wave function [48]

$$\Psi\left(\mathbf{x}_1^+, \cdots, \mathbf{x}_{\mathrm{N}_+}^+\right) = \tilde{\mathcal{F}}\left(\overline{z}_1^{\,+}, \cdots, \overline{z}_{N_+}^{\,+}\right) \prod_{i<j} \left(z_{\mathrm{i}}^+ - z_{\mathrm{j}}^+\right)^{-\frac{1}{k}} \quad . \qquad (V.8)$$

This wave function acquires the anyonic phase $\exp\left(\frac{i\pi}{k}\right)$ after the exchange of two particles.

Then, if we consider the model (V.2) as a quantum version of the magnetic bubble system, we see that it behaves as a quantum anyon system. Furthermore, the wave function (V.8) can be employed to describe a condensate state of bosonic solitons and may be related to the quantum disordered state of the original ferromagnet [49]. Actually, the previous wave function is not normalizable on the plane. Its normalization can be obtained by introducing an external magnetic field in our topological model. Normalizable eigenstates can be obtained also on a compact surface Σ without introducing an external field [61]. However, in such a case the anyon gauge does not seem very fruitful.

VI Conclusions

We have studied the classical non-abelian CS model, where the gauge fields satisfy certain geometrical requirements. In particular, if the gauge fields belong to a Z_2-graded Lie algebra, we are led in a natural way to a decomposition of the action, in which a set of matter fields interacts via a (generally non-abelian) CS field. The models, described by the action with this property, are compared with the systems arising from the so-called tangent space representation of the generalized Heisenberg models in 2+1 dimensions. From this analysis we see that some special planar spin models can be obtained by a generally non-abelian CS theory via a convenient gauge-fixing condition. For example, both the well-known $SU\left(2\right)$ Heisenberg model and the Ishimori model, which is completely integrable, can be associated with the TFT in special gauges. The symplectic structure of these models has been investigated mainly within the gauge-invariant procedure. Thus, we can provide also the quantum theory of the models. In particular, the many-body wave function is multivalued, since in the anyonic gauge it gets the phase $\exp\left(\frac{i\pi}{k}\right)$ by exchanging two particles. The corresponding anyons have spin $s = \frac{1}{2k} \left(\mathrm{mod}\ \mathbf{Z}\right)$.

Now, our approach to TFT needs further developments and several interesting applications are in order.

First of all, the possibility to handle nonlinear completely integrable systems in the context of the quantum TFT is not completely exploited. A suitable BRST approach has to be applied in order to restore the general covariance. However, our concern is that of finding new results, or also old results in a more easy way, just working with integrable structures.

Since the models discussed in this paper come from a quite abstract framework, it is interesting to see whether some of them play any special role in the domains of planar physics, where non-local interactions of the CS type become important. At present, we notice only that in our models the coupling constants of the theory are fixed by the geometry of the original non-abelian pure CS model and by the constant k, which is an integer from the quantum theory. Thus, our models are special cases of many other CS-gauge field theories (see [11]- [15]), in which the above mentioned coupling constants can take arbitrary real values and, for this reason, can be considered as pertubations or deformations of ours.

Other possible physical applications of these ideas concern the description

of the multi-layer Hall systems, treated as planar pseudospin ferromagnets [62] and the study of the quantum Hall fluids in terms of boundary excitations and of CS topological field theories [63].

Furthermore, since the CS TFT is exactly solvable in the $SU(2)$ case for any three-manifolds [2], we speculate that our procedure should lead to a field theoretical description of an anyonic system on an arbitrary Riemann surface Σ (sphere, torus and higher genus surfaces). In these cases the operator ∂_j^{-1} must be modified, in order to include a residual topological interaction [64]. However, when these systems are obtained by the described procedure, their solvability is assured by the above mentioned general property.

It is also physically interesting to consider the extension of our procedure to noncompact groups. In fact, for the $ISO(2,1)$ group the corresponding model is equivalent to the TFT studied by Witten in connection with the (2+1)-dimensional quantum gravity [65].

Furthermore, our procedure can be used for classifying classical topological field models by the point of view of their integrability property. This corresponds to classify the supplementary constraints, by requiring that the system of equations (II.18) - (II.23) and (II.24) allows a Lax pair, in analogy with the procedure and the results contained in [66]. It is remarkable that in our scheme, either integrable and non-integrable gauge-fixing conditions are found. The question of the existence of a relationship between these two type of structures in the unifying framework supplied by the original non-abelian CS model is still open. A hint for solving this problem may be found in the gauge transformations generated by $\lambda^{(1)}$, which mixes the components of the chiral current by the relations

$$\begin{aligned} \delta J^{(0)} &= [J^{(1)}, \lambda^{(1)}] \\ \delta J^{(1)} &= [J^{(0)}, \lambda^{(1)}] + d\lambda^{(1)} \quad . \end{aligned} \tag{VI.1}$$

We notice that the transformations (VI.1) imply a non trivial mixing among old matter and gauge fields.

Finally, we point out that in a similar fashion one can build up more general models, by considering non-symmetric spaces. For instance, models involving N abelian CS fields can be constructed in a reductive homogeneous space $\mathcal{G}\ /\mathcal{H}$, where $\mathcal{H} = U(1)^N$ [26, 27]. Another possible generalization can be obtained just considering an higher dimensional $SU(n)$ group. In this case we can solve the GCS laws related to the maximal abelian subgroup of

local $U(1)$ gauge symmetries. These theory should be solved in terms of the Toda models, at least for the self-dual reduction [16].

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